Optimal control of discharge from a reservoir based on Kushner's finite Markov chain approximation of river flow diffusion process

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Résumé:

On etudie ici le problème de la décharge optimale de l'eau d'un réservoir, le barrage réservoir Seine, par une méthode de la théorie du contrôle optimal stochastique. Les éléments constituants la systeme sont le stock S_t de l'eau dans le réservoir, le débit naturel Y_t dans la rivière (Seine), la lâchure(décharge) W_t de la réservoir et la demande D_t. On cherche le processus $\{W_t\}$ de la lâchure qui minimise la espérance de la intégrale d'une fonction (coût) de la différence(déficit) entre la demande D_t et le debit total $Y_t + W_t$. Le processus de la demande est supposé que deterministe. Le processus du débit naturel est identifié comme un processus de diffusion, et on utilise la méthode de Kushner qui reduit les problèmes contrôles des processus des diffusions à ceux des chaînes markov finies. Supposant la loi des utilités marginales decroissants sur le coût, la lâchure optimale est obtenu que le point equilibré où la utilité marginale prospective et celle instntanée sont balancés. Les résultats numeriques nous fournissent beacoup de renseignements utiles sur la gestion pratique du réservoir.

Abstract:

We study here the problem of optimal discharge of water from a reservoir, the reservoir Seine dam by a method of the optimal stochastic control theory. The elements constituting the system are water stock S_t in the reservoir, natural flow Y_t in the river (Seine)

N.B. This study was done during the author's stay at INRIA in the research group of M. Alain BENSOUSSAN with the collaboration of M. DELEBECQUE.

water stock S_t in the reservoir, natural flow Y_t in the river (Seine), discharge W_t from the reservoir and demand D_t. We search for process W_t of the discharge which minimizes the expectation of the integral of a function (cost) of the difference (shortage) between demand D_t and total flow Y_t and W_t. The demand process is assumed to be deterministic. The natural flow process is identified as a diffusion process, and we use the method of Kushner which reduces the control problems of diffusion processes to those of finite Markov chains. Assuming the low of diminishing marginal utilities on the cost, the optimal discharge is obtained as the equlibrium point where the prospective marginal utility and the instantaneous one are balanced. The numerical results offers us much useful information on the control of the reservoir.

OPTIMAL CONTROL OF DISCHARGE FROM A RESERVOIR BAESD ON KUSHNER'S FINITE MARKOV CHAIN APPROXIMATION OF RIVER FLOW DIFFUSION PROCESS

1. INTRODUCTION

In this research, we treat a control problem of discharge from a reservoir facining the (river) Seine called "le barrage reservoir Seine" on the basis of statistical properties of the river natural flow obtained from the data in the past three decades. The reservoir is used to supply water in accordance with the deficiency of the flow to the industrial and the agricultural demand(cf. Fig. 1.1)[1].

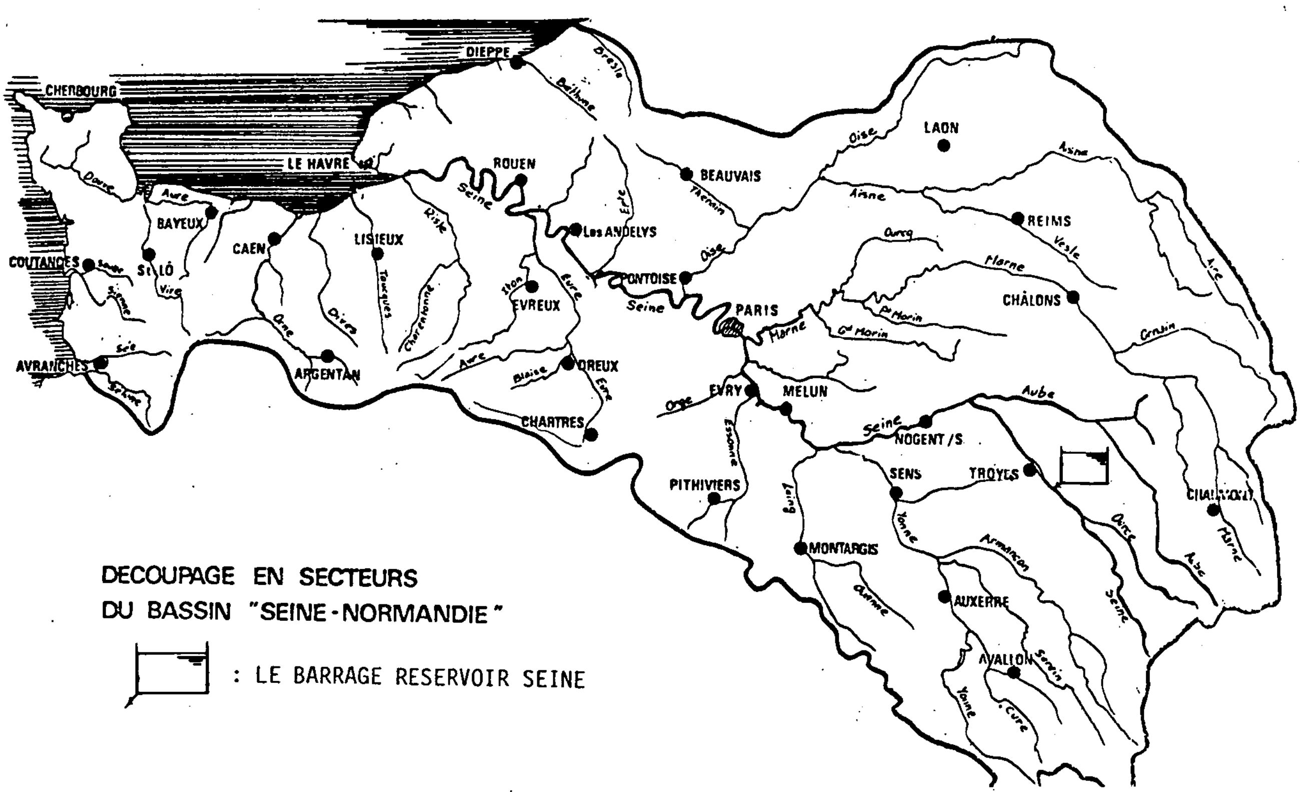

Fig. 1.1 The site of the reservoir.

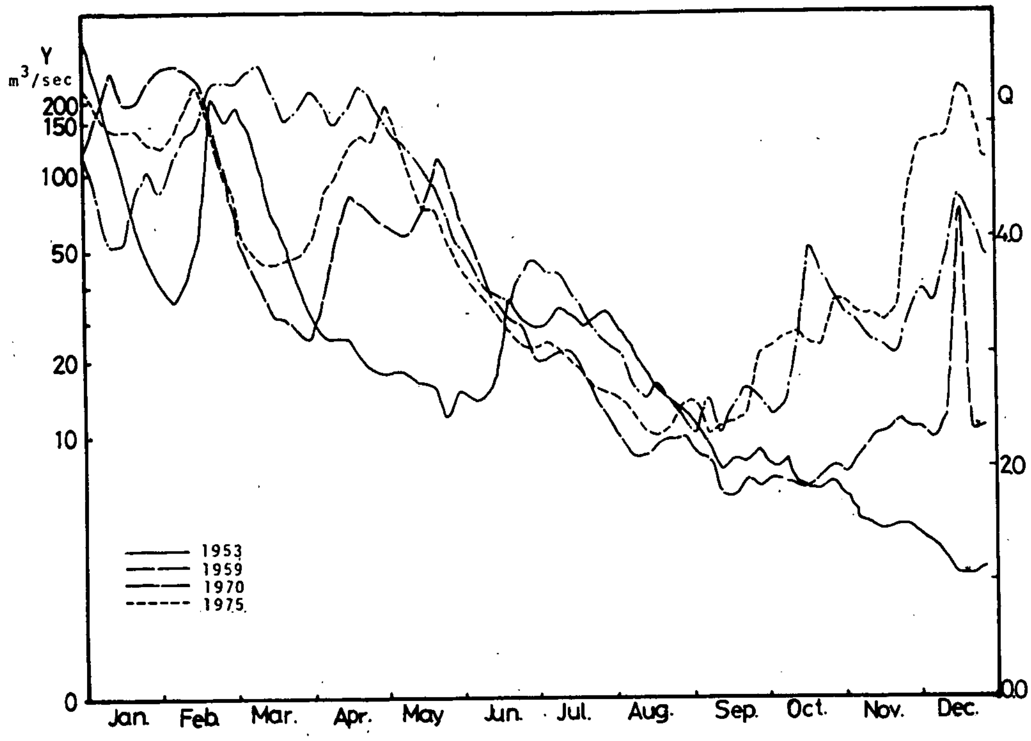

Fig. 1.2 Some typical behavior of natural flow in the Seine.

Fig. 1.2 shows some typical behavior of natural flow in the upper stream of Paris in the past three decades. It is shown in all of the cases that the flow in the first half of the year (i.e. from Jan. 1st to May 30th) is sufficiently large enough to cover the demand. In the second half of the year (from June 1st to Dec. 31st), however, the natural flow is insufficient to cover the demand; it necessitates the discharge from the reservoir. That is, the former half can be regarded as the time period for pouring water to the reservoir and the latter half is the time period for discharging water from the reservoir. A more precise examination of the data leads to a view that the latter period is consisted of two different periods; the flow in the former one (June 1st to Sept. 30th) is mainly consisted of snow water and the flow of the latter one (from Oct. 1st to Dec. 31st) is consisting of rain water. They have quite different characteristics; the behavior of the snow water flow is rather regular and has decreasing tendency, while that of the rain water flow contains many irregular jumps.

Thus, we face to two different approaches to the control problem. One way is to identify the flow in the total period, i.e. from the 1st of June to the 31st of December by a single model and then search for the optimal discharge policy for the whole period. The other way is to decompose the problem into two subproblems: the optimal discharge problem in the former subperiod (from June 1st to Sept. 30th) and the problem in the later subperiod (from Oct. 1st to Dec. 31st). In the second approach, it is necessary to solve the latter subproblem at first, and then we can proceed to the former subproblem.

The identification of the natunal flow in the whole control period needs much elaboration and data, for it necessitates unifying two quite different characteristics in the data. One feasible way is to constract a diffusion process model with jumps.[2] Another way is to make a Markov chain model by discretizing the flow (in the data) into finite flow levels (states) and then by estimating the transition probabilityies as the frequencies of the occurrence of corresponding transitions in the data. These identifications processes, however, need much amount of data in order to provide a sufficient basis for the solution of the control problem. The data available are, as aforementioned, quite limited, i.e. the daily natural flow from 1950 to 1976. Hence, we will adopt the second approach.

Because of the irregularity in the flow data in the latter subperiod (from Oct. 1st to Dec. 31st), we will have the same difficulty as above in the identification in this period. Thus, we will not pursue here the optimal discharge problem in the period, but instead we presume several plausible results of the problem and then search for the optimal discharge policy for the former subperiod (from June 1st to Sept. 30th).

The objective of this research is to clarify the effect of various factors such as statistical properties of natural flow, supply-demand gap etc. on the control policy and to obtain beneficial information for the revision of the actural discharge policy.

There exist various models of natural flow in rivers depending of the period, frequency, precision and objective.[3] As commonly understood, daily flow has very many irregularities causing difficulty in its identification. Also it is well-known that mean flow in several days, say in a weak, belongs to a logarithmic diffusion process, i.e., (natural) logarithm of the mean flow can be identified as a diffusion process.[4] As disussed in the sequel, we will identify the flow on the basis of 5 day mean flow by the use of linear regression analysis.[5] The period 5 days seems to provide a compromising point between the precision of modelling and the accuracy of discharge policy.

2. SETTING OF THE OPTIMAL DISCHARGE PROBLEM FROM THE RESERVOIR AS A CONTROL PROBLEM

The region with which the reservoir "le barrage reservoir Seine" is concerned is widespread, and the insufficiency of natural flow may vary according to the area.[1] Thus, the supply-demand gap, which should be reflected in the evaluation of discharge policy, must be evaluated at several points in the region. We presume, however, that the gap can be evaluated at a single (standard) point. We also simplify the problem by disregarding the effect of pouring water into the reservoir in the discharge period.

By these simplifications, the whole system consists of four variables: stock S_t in the reservoir at time t, amount of discharge W_t, natural flow Y_t and demand D_t. The unit we will use is "m^3/sec" for flow and discharge, and "day" for time. Therefore, S_t will be measured by $m^3/sec \times 1$ day $(= 8.64 \times 10^4\ m^3)$, and the other variables W_t, Y_t and D_t are measured by m^3/sec. (cf. Fig. 2.1).

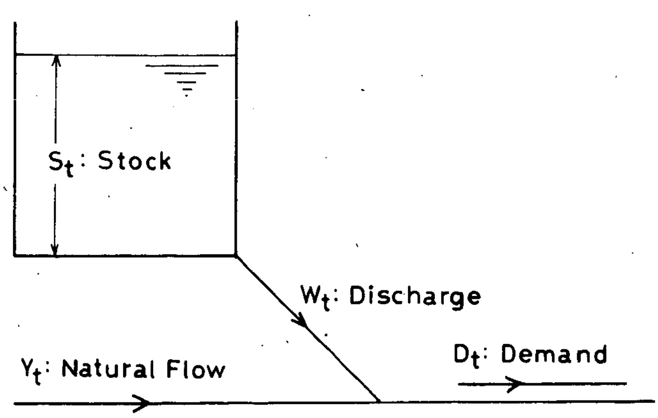

Fig. 2.1 Setting of the discharge problem.

Demand D_t is dependent on various factors such as agricultural, industrial, atmospheric factors etc., and hence $\{D_t\}$ should be regarded as a stochastic process. Simplifying the problem, however, we presume that $\{D_t\}$ is a deterministic process, and hence the state variables in this problem are S_t, W_t and Y_t.

The cost function measuring the effectiveness of discharge policy $\{D_t\}$ is assumed to have the following integral form.

$$V = V(T_0, Y_{T_0}, S_{T_0})$$

$$\triangleq E[\int_{T_0}^{T_1} C'(t, D_t, Y_t, W_t)dt + V(T_1, Y_{T_1}, S_{T_1}) \mid T_0, Y_{T_0}, S_{T_0}]$$

$$T_0 : \text{June 1st}, \quad T_1 : \text{Sept. 30th} \tag{2.1}$$

Term $V(T_1, Y_{T_1}, S_{T_1})$ in the right-hand side evaluates the cost of having stock S_{T_1} at the final time T_1 provided that the natural flow is Y_{T_1}. This evaluation should be obtained by solving the similar optimal discharge problem for time period $[T_1, T_2]$ (T_2 : Dec. 31st). However, as aforementioned, we presume several resonable setting of the evaluation beforehand, which will be clarified in section 6.

Also, as mentioned earlier, $\{Y_t\}$ is assumed to be a logarithmic diffusion process. Thus, the system equation in this problem is given as follows:

$$dS_t / dt = -W_t \tag{2.2}$$

$$\Delta Q_t = f(Q_t, t)\Delta t + \sqrt{a(Q_t, t)}\Delta B_t \tag{2.3}$$

$$Q_t \triangleq \ln Y_t \tag{2.4}$$

$$\{B_t\} : \text{Wiener Process} \tag{2.5}$$

We search for the optimal discharge process $\{W_t\}(T_0 \le t \le T_1)$ which minimizes cost (2.1) subject to the next constraint conditions.

$$S_t \ge 0 \quad \text{for} \quad T_0 \le t \le T_1 \tag{2.6}$$

$$0 \le W_t \le W_{max} \quad \text{for} \quad T_0 \le t \le T_1 \tag{2.7}$$

3. IDENTIFICATION OF NATURAL FLOW

We will construct a model of 5 day mean flow process $\{\overline{Y}_t\}$, i.e.,

$$\overline{Y}_t \triangleq (1/5)\ (Y_t + Y_{t+1} + Y_{t+2} + Y_{t+2} + Y_{t+3} + Y_{t+4}), \tag{3.1}$$

by tranforming it into (natural) logarithmic value

$$Q_t \triangleq \ln Y_t. \tag{3.2}$$

The diffusion process model (2.3) of Q_t is given on the basis of the regression analysis as follows:

$$Q_{t+\Delta t} = \alpha_t + \beta_t \cdot Q_t + \varepsilon_t$$

$$\text{for}\quad T_0 \le t \le T_1,\quad \Delta t : 5\text{days}, \tag{3.3}$$

where it is assumed that

$$E[\varepsilon_t] = 0 \qquad \text{for}\quad T_0 \le t \le T_1 \tag{3.4}$$

$$E[\varepsilon_t \cdot Q_t] = E[\varepsilon_t] \cdot E[Q_t] = 0 \qquad \text{for}\quad T_0 \le t \le T_1 \tag{3.5}$$

From these assumptior , it follows that

$$\alpha_t = m_{t+1} - \beta_t \cdot m_t, \tag{3.6}$$

$$\beta_t = \sigma_{t+\Delta t,t} / \sigma_{t,t}, \tag{3.7}$$

where

$$m_t \triangleq E(Q_t) \tag{3.8}$$

$$\sigma_{t,t'} \triangleq E[(Q_{t'} - m_{t'})(Q_t - m_t)] \tag{3.9}$$

From (3.3), we have

$$\Delta Q_t \triangleq Q_{t+\Delta t} - Q_t$$

$$= \alpha_t + (\beta_t - 1)Q_t + \varepsilon_t \tag{3.10}$$

Hence, coefficients f and a in diffusion model (2.3) are obtained as

$$f(Q_t, t) = [\alpha_t + (\beta_t - 1)\cdot Q_t] \cdot (\Delta t)^{-1} \tag{3.11}$$

$$a(Q_t, t) = E[\varepsilon_t^2] \cdot (\Delta t)^{-1} \tag{3.12}$$

It should be noted that drift term coefficient f is dependent linearily on (logarithmic) flow Q, while a is only dependent on t(cf. Fig. 3.1)

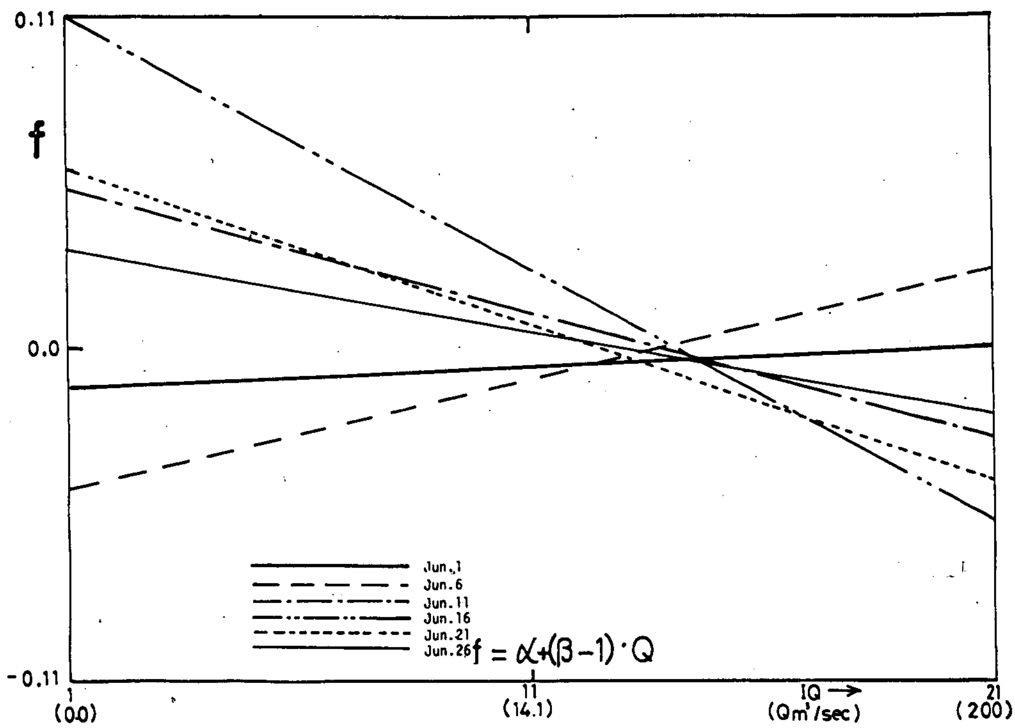

Fig. 3.1 Dependence of coefficient f upon Q.

The data used in the above identification process is the mean flow data $\{\bar{Y}_t\}$ in 26 years, i.e. from 1950 - 1976.

4. SOLUTION OF THE OPTIMAL CONTROL PROBLEM BY KUSHNER'S MARKOV CHAIN APPROXIMATION METHOD

As Q_t being substituted for natural flow Y_t in the above identification, we will regard cost function $V(T_0, Y_{T_0}, S_{T_0})$ as function of T_0, Q_{T_0} and S_{T_0}, i.e.,

$$V = V(T_0, Q_{T_0}, S_{T_0})$$

$$\triangleq \inf_{\substack{\{W_t\} \\ T_0 \le t \le T_1}} \{ E [\int_{T_0}^{T_1} C(t, D_t, Q_t, W_t)dt + V(T_1, Q_{T_1}, S_{T_1}) \mid T_0, Q_{T_0}, S_{T_0}]\}, \quad (4.1)$$

where

$$C(t, D_t, Q_t, W_t) \triangleq C'(t, D_t, e^{Q_t}, W_t)$$

$$\text{for} \quad T_0 \le t \le T_1 \tag{4.2}$$

For an arbitrary time $t(T_0 \le t \le T_1)$, flow Q and stock S, we define cost function $V(t, Q, S)$ as

$$V(t, Q, S) \triangleq \inf_{\{W_u\}_{t \le u \le T_1}} \{ E [\int_t^{T_1} C(u, D_u, Q_u, W_u) du$$

$$+ V(T_1, Q_{T_1}, S_{T_1}) \mid t, Q, S] \} \tag{4.3}$$

From (2.2) - (2.7), it can be readily seen that the above function satisfies the next equality:

$$V_t + f \cdot V_Q + (1/2) \cdot V_{QQ} + \inf_{0 \le W \le W_{max}} \{ (-W) \cdot V_S + c(t, D_t, Q_t, W_t) \} = 0$$

$$\text{for} \quad T_0 \le t \le T_1 \tag{4.4}$$

In Kushner's method, the above partial derivatives are approximated as[6]

$$V_t \cong (V(t + \Delta t, Q, S) - V(t, Q, S)) / \Delta t \tag{4.5}$$

$$V_Q \cong (V(t + \Delta t, Q + \Delta Q, S) - V(t + \Delta t, Q, S)) / \Delta Q \quad \text{if} \quad f(t, Q, S) \ge 0$$

$$\cong (V(t + \Delta t, Q, S) - V(t + \Delta t, Q - \Delta Q, S)) / \Delta Q \quad \text{if} \quad f(t, Q, S) < 0 \tag{4.6}$$

$$V_{QQ} \cong (V(t + \Delta t, Q + \Delta Q, S) + V(t + \Delta t, Q - \Delta Q, S) - 2V(t + \Delta t, Q, S)) / (\Delta Q)^2 \tag{4.7}$$

$$V_S \cong (V(t + \Delta t, Q, S) - V(t + \Delta t, Q, S - \Delta S)) / \Delta S \tag{4.8}$$

From the above approximations and (4.4), we obtain the next renewal equation for cost function V.

$$V(t, Q, S) = \inf_{0 \le W \le W_{max}} V^W(t, Q, S) \tag{4.9}$$

$$V^W(t, Q, S) \triangleq P_t^W(Q, S \mid Q, S) \cdot V(t + \Delta t, Q, S)$$
$$+ P_t^W(Q + \Delta Q, S \mid Q, S) \cdot V(t + \Delta t, Q + \Delta Q, S)$$
$$+ P_t^W(Q - \Delta Q, S \mid Q, S) \cdot V(t + \Delta t, Q - \Delta Q, S)$$
$$+ P_t^W(Q, S - \Delta S \mid Q, S) \cdot V(t + \Delta t, Q, S - \Delta S)$$
$$+ C(t, D_t, Q_t, W_t) \cdot \Delta t, \qquad (4.10)$$

where

$$\begin{aligned} P_t^W(Q, S \mid Q, S) &\triangleq 1 - (|f| / \Delta Q + a_t / (\Delta Q)^2 + W_t / \Delta S) \cdot \Delta t \\ P_t^W(Q + \Delta Q, S \mid Q, S) &\triangleq (f^+ / \Delta Q + (1/2) \cdot (a_t / (\Delta Q)^2)) \cdot \Delta t \\ P_t^W(Q - \Delta Q, S \mid Q, S) &\triangleq (f^- / \Delta Q + (1/2) \cdot (a_t / (\Delta Q)^2)) \cdot \Delta t \\ P_t^W(Q, S - \Delta S \mid Q, S) &\triangleq (W_t / \Delta S) \cdot \Delta t \end{aligned} \qquad (4.11)$$

$$\begin{aligned} f^+ &\triangleq \max(0, f) \\ f^- &\triangleq -\min(0, f) = \max(0, -f) \end{aligned} \qquad (4.12)$$

The above formulas show that the control problem of the system prescribed by (2.2) and diffusion process (2.3) can be reduced to a control problem of a Markov chain with transition probabilities (4.11) which are dependent on control variable W_t. This chain is specific in the sense that only the transitions between neighbouring states are possible.

The optimal discharge $W_t(Q, S)$ at time t under natural flow Q and stock S is given as the value of attainning the minimum of $V^W(t, Q, S)$.

$$V^{W_t^*(Q, S)}(t, Q, S) = \min_{0 \le W \le W_{max}} V^W(t, Q, S)$$
$$= V(t, Q, S) \qquad (4.13)$$

Unit time Δt of control must satisfy the inequality below, for the probabilities in (4.11) are nonnegative.

$$\Delta t \le (|f| / \Delta Q + a / (\Delta Q)^2 + W_{max} / \Delta S)^{-1} \qquad (4.14)$$

5. INTERPRETATION OF THE OPTIMAL DISCHARGE AS THE EQUILIBRIUM POINT OF SUPPLY (PROSPECTIVE) AND DEMAND (CURRENT) UTILITIES

The Markov chain approximation (4.10) and (4.11) can be rewritten as

$$
\begin{aligned}
V^W(t, Q, S) = {} & P_t(Q \mid Q) \cdot V(t + \Delta t, Q, S) \\
& + P_t(Q + \Delta Q \mid Q) \cdot V(t + \Delta t, Q + \Delta Q, S) \\
& + P_t(Q - \Delta Q \mid Q) \cdot V(t + \Delta t, Q - \Delta Q, S) \\
& + (W \cdot \Delta t) \cdot ((V(t + \Delta t, Q, S - \Delta S) \\
& \qquad - V(t + \Delta t, Q, S)) \ / \ \Delta S \\
& + C(t, D_t, Q, W) \cdot \Delta t \\
= {} & E_Q [V(t + \Delta t, Q, S) \mid t, Q] \\
& + (W \cdot \Delta t) \cdot ((V(t + \Delta t, Q, S - \Delta S) - V(t + \Delta t, Q, S))/\Delta S) \\
& + C(t, D_t, Q, W) \cdot \Delta t,
\end{aligned}
\tag{5.1}
$$

where P_t's are defined as

$$
\begin{aligned}
P_t(Q \mid Q) &= 1 - (|f| / \Delta Q + a / (\Delta Q)^2) \cdot \Delta t \\
P_t(Q + \Delta Q \mid Q) &= (f^+ / \Delta Q + (1/2) \cdot (a / (\Delta Q)^2)) \cdot \Delta t \\
P_t(Q - \Delta Q \mid Q) &= (f^- / \Delta Q + (1/2) \cdot (a / (\Delta Q)^2)) \cdot \Delta t
\end{aligned}
\tag{5.2}
$$

and are regarded to be transition probabilities from state (flow level) Q to Q, $Q + \Delta Q$, and $Q - \Delta Q$, respectively, when the diffusion of natural flow from flow level Q is condensed into three flow levels $Q - \Delta Q$, Q, and $Q + \Delta Q$; and E_Q represents the expectation with respect to the probabilities. Hence, from (4.13), the optimal discharge $W_t^*(Q, S)$ is the value of W which minimizes the following quantity:

$$
\begin{aligned}
& (W \cdot \Delta t) ((V(t + \Delta t, Q, S - \Delta S) - V(t + \Delta t, Q, S)) / \Delta S) \\
& \quad + C (t, D_t, Q, W) \cdot \Delta t
\end{aligned}
\tag{5.3}
$$

Let us introduce the quantity

$$
U (S \mid t, Q) \triangleq V (t, Q, 0) - V (t, Q, S)
\tag{5.4}
$$

and call it the "total utility" of stock S. Also, we introduce

$$R(W\cdot\Delta t \mid t, Q) \triangleq C(t, D_t, Q, 0)\Delta t - C(t, D_t, Q, W)\Delta t \tag{5.5}$$

and call it the total utility of discharged water whose amount is $W\cdot\Delta t$.

From definition (4.2), (5.4) and (5.5), $U(s \mid t, Q)$ can be rewritten as

$$\begin{aligned} U(S \mid t, Q) &= \sup_{\{W_u\}} [\sum_{t\le u\le T_1} R(W_u\cdot\Delta t \mid u, Q) \\ &\quad - V(T_1, Q_{T_1}, S_{T_0} - \sum W_u \cdot \Delta t)] \\ &\quad + V(T_1, Q_{T_1}, 0), \end{aligned} \tag{5.6}$$

where $\sum$ stands for the summation over time intervals $[u, u + \Delta t]$'s in period $[t, T]$. Thus, $R(W\cdot\Delta t \mid t, Q)$ can be interpreted as the instantaneous (current) utility of discharging water $W\cdot\Delta t$ at time t under natural flow Q, and U can be regarded as the accumulation of these (prospective) utilities up to the final time T_1 under the optimal discharge policy.

When we regard discharging water from a reservoir as an act of selling water from the supplyer (the reservoir) to a consumer, this "prospective utility" U and the "current utility" R will be interpreted as the "supply-side total utility" of stock S and the "demand-side total utility" of discharged water $W\cdot\Delta t$, respectively.

The derivatives of these quantities by unit volume of water are called "marginal utilities", i.e.,

$$\begin{aligned} u_S(t, Q) &\triangleq \frac{\partial U(S \mid t, Q)}{\partial S} \quad : \text{supply-side marginal utility} \\ &\cong (V(t, Q, S) - V(t, Q, S))/\Delta S \end{aligned} \tag{5.7}$$

$$r_W(t, Q) \triangleq \frac{\partial R(W\cdot t \mid t\cdot Q)}{\partial (W\cdot\Delta t)} : \text{demand-side marginal utility} \tag{5.8}$$

In most of the cases, the following law on r_W called "law of diminishing marginal utilities" can be assumed to hold.

$$r_W(t, Q) \;:\; \text{monotonically non-increasing w.r.t. } Q \tag{5.9}$$

In this case, it will be readily seen that $W_t^*(Q, S)$, the minimizing value W of (5.3) subject to (2.6) and (2.7) is given as

$$W_t^*(Q, S) = \hat{W} \qquad \text{if} \quad 0 \le \hat{W} \le W'_{max}$$

$$W_t^*(Q, S) = 0 \qquad \text{if} \quad \hat{W} < 0 \tag{5.10}$$

$$W_t^*(Q, S) = W_{max} \qquad \text{if} \quad W'_{max} < \hat{W}$$

where W'_{max} is given as

$$W'_{max} = \min(W_{max}, S/\Delta t) \tag{5.11}$$

and $\hat{W}$ is the solution of next equation.

$$r_{\hat{W}}(t, Q) = u_S(t + \Delta t, Q) \tag{5.12}$$

Namely, the optimal discharge W_t^* is prescribed as the equilibrium point $\hat{W}$ of demand-side marginal utility r_W and supply-side marginal utility u_S as illustrated in Fig. 5.1.

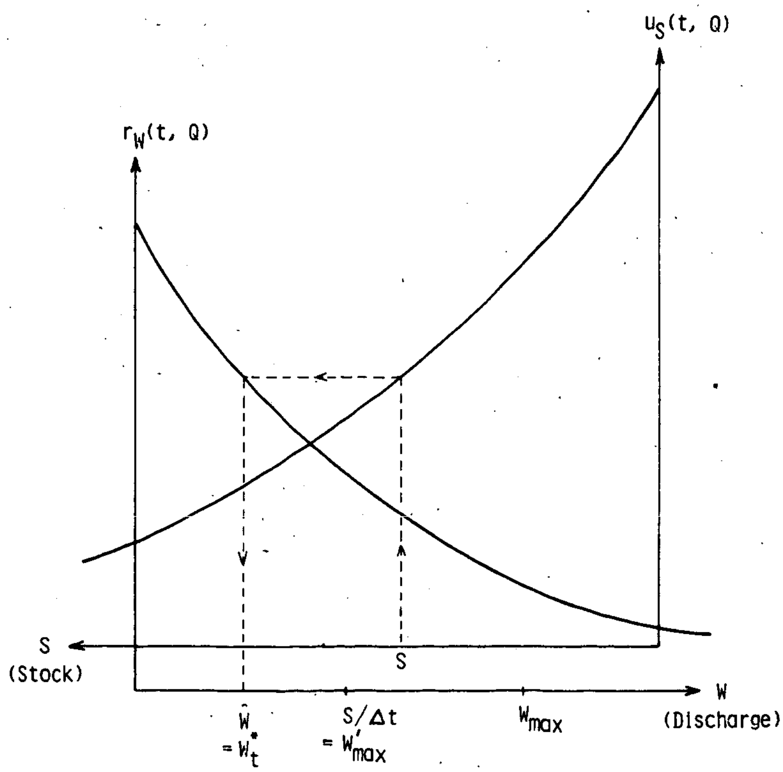

Fig. 5.1 Optimal discharge W_t^* as the equilibrium point of marginal utilities r_W and u_S.

By the use of optimal discharge W_t^*, the renewal equation (4.9) of V is rewritten as

$$V(t, Q, S) = V^{W_t^*(Q, S)}(t, Q, S) \tag{5.13}$$

It is obvious from (5.10) and (4.10) that

$$\begin{aligned} V(t, Q, 0) &= V^0(t, Q, S) \\ &= P_t(Q \mid Q) \cdot V(t + \Delta t, Q, 0) \\ &\quad + P_t(Q + \Delta Q \mid Q) \cdot V(t + \Delta t, Q + \Delta Q, 0) \\ &\quad + P_t(Q - \Delta Q \mid Q) \cdot V(t + \Delta t, Q - \Delta Q, 0) \\ &\quad + C(t, D_t, Q, 0)\Delta t \end{aligned} \tag{5.14}$$

Hence, from (5.4), (5.5) and (5.6), we have

$$\begin{aligned} U(S \mid t, Q) &= P_t(Q \mid Q) \cdot U(S \mid t + \Delta t, Q) \\ &\quad + P_t(Q + \Delta Q \mid Q) \cdot U(S \mid t + \Delta t, Q + \Delta Q) \\ &\quad + P_t(Q - \Delta Q \mid Q) \cdot U(S \mid t + \Delta t, Q - \Delta Q) \\ &\quad - u_S(t + \Delta t, Q) \cdot (W_t^*(Q, S) \cdot \Delta t) \\ &\quad + R(W_t^*(Q, S)\Delta t \mid t, Q) \\ &= E_Q[U(S \mid t + \Delta t, Q) \mid t, Q] \\ &\quad - u_S(t + \Delta t, Q) \cdot (W_t^*(Q, S) \cdot \Delta t) \\ &\quad + R(W_t^*(Q, S) \cdot \Delta t \mid t, Q) \end{aligned} \tag{5.15}$$

which, together with (5.6), yields the next renewal equation on the marginal utility for supply-side.

$$u_S(t, Q) = E_Q[u_S(t + \Delta t, Q) \mid t, Q] + \Delta u_S(t, Q), \tag{5.16}$$

where $\Delta u_S(t, Q)$ is defined and calculated as follows:

$$\begin{aligned} \Delta u_S(t, Q) \triangleq \{ & R(W_t^*(Q, S) \cdot \Delta t \mid t, Q) - u_S(t + \Delta t, Q) \cdot \\ & (W_t^*(Q, S) \cdot \Delta t) \quad - R(W_t^*(Q, S - \Delta S) \cdot \Delta t \mid t, Q) \\ & - u_{S-\Delta S}(t + \Delta t, Q)(W_t^*(Q, S) \cdot \Delta t)\} \end{aligned}$$

$$= \int_0^{W_t^*(Q.\ S)} (r_W(t, Q) - u_S(t + \Delta t, Q))\, d(W \cdot \Delta t)$$

$$- \int_0^{W_t^*(Q,\ S-\Delta S)} (r_W(t, Q) - u_{S-\Delta S}(t + \Delta t, Q))\, d(W \cdot \Delta t)$$

$$= \{u_{S-\Delta S}(t + \Delta t, Q) - u_S(t + \Delta t, Q)\} \cdot W_t^*(Q, S - \Delta S) \cdot \Delta t$$

$$+ \int_{W_t^*(Q,\ S-\Delta S)}^{W_t^*(Q,\ S)} (r_W(t, Q) - u_S(t + \Delta t, Q))\, d(W \cdot \Delta t) \tag{5.17}$$

Namely, Δu_S is the sum of relative benefits of discharging $W_t^*(Q, S) \cdot \Delta t$ of water from stock S, which will be examined in the next section.

6. RESULTS AND DISCUSSIONS

During control period $[T_0, T_1]$ (T_0: the 1st of June, T_1: the 30th of September), demand process $\{D_t\}$ is assumed to be deterministic, as aforementioned, and to decrease linearly in the first two months, June and July, and to be constant in the second two months, August and September, as depicted in Fig. 6.1.

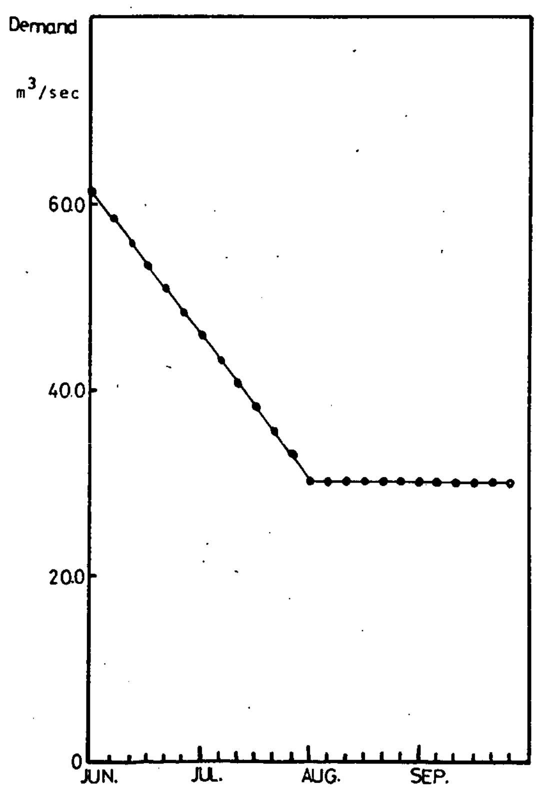

Fig. 6.1 Demand process in the control period, June - September.

Also according to the data given, we presume the minimum and the maximum value of (logarithmic) natural flow Q(= ln Y) as

$$Q_{min} = 0.0 \qquad Q_{max} = l_n\ 200.0 = 5.298 \qquad (m^3/sec) \tag{6.1}$$

Let us discretize the state space $[Q_{min}, Q_{max}]$ of the diffusion process of Q into finite (representative) states (flow levels):

$$Q_1 = Q_{min},\ Q_2 = Q_1 + \Delta Q,\ \cdots,\ Q_n = Q_{max} = Q_{n-1} + \Delta Q \tag{6.2}$$

$$\Delta Q = (Q_{max} - Q_{min}) \ / \ (n - 1) \tag{6.3}$$

Then probabilities P_t's given by (5.2) can be interpreted as the transition probabilities of the Markov chain with state space $\{Q_1, Q_2, \cdots, Q_n\}$ which approximates the original diffusion process prescribed by (2.3), (3.11) and (3.12). As illustrated in Fig. 6.2, The

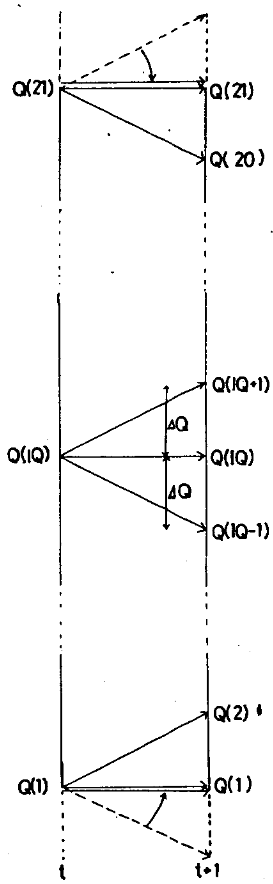

Fig. 6.2 Markovian transition structure in natural flow Q.

transition probability leading the outside of the state space $[Q_{min}, Q_{max}]$ from extremal state Q_{min} (or Q_{max}) is added to the transition probability of staying at Q_{min} (Q_{max}) so as to the process being inside the state space.

Similarly, let the space $[0, S_{max}]$ of stock S be discretized into m states (stock levels) such as

$$S_1 = 0,\ S_2 = \Delta S,\ S_3 = S_2 + \Delta S,\ \cdots,\ S_m = S_{max} = S_{m-1} + \Delta S \quad (6.4)$$

$$\Delta S = S_{max} \ / \ (m - 1) \quad (6.5)$$

The reservoir, le barrage reservoir Seine, has the capacity of $1.6 \times 10^8\ m^3$ which is measured, by the unit $m^3/sec \times 1$ day, as

$$S_{max} = 1852.0 \quad (m^3 / sec \times 1\ day) \quad (6.6)$$

In this research, we use

$$n = m = 21 \quad (6.7)$$

which yield

$$\Delta Q = 5.298 \ / \ 20 = 0.2649 \quad (6.8)$$

$$\Delta S = 1852.0 \ / \ 20 = 92.6 \quad (6.9)$$

The corresponding flow level Y_i of logarithmic flow level Q_i for i = 1, 2, $\cdots$, 21 is given in Table 6.1. The maximum value of discharge is set as

$$W_{max} = 40.0 \quad (m^3 / sec)$$

i	Q_i	Y_i	i	Q_i	Y_i
1	0.0	1.0	12	2.914	18.4
2	0.265	1.303	13	3.179	24.0
3	0.530	1.699	14	3.444	31.3
4	0.795	2.214	15	3.709	40.8
5	1.057	2.886	16	3.974	53.1
6	1.325	3.760	17	4.239	69.3
7	1.589	4.901	18	4.504	90.3
8	1.854	6.388	19	4.768	117.7
9	2.119	8.325	20	5.033	153.5
10	2.384	10.850	21	5.298	200.0
11	2.649	14.140			

Table 6.1 Setting of flow levels Q_i and Y_i for i = 1, 2, $\cdots$, 21.

The above parameters, the regression analysis in section 3, and (4.14) lead us to set unit time Δt of control as

$$\Delta t = 1 \qquad (\text{day}) \tag{6.10}$$

which is shorter than the unit time (5 days) used in the identification of natural flow, i.e., the unit time in diffusion model (2.3). We will assume that the diffusion model also applies in this short unit time case without modifying coefficients f or a. The (instantaneous) cost function evaluating the loss due to the demand-supply gap is set as

$$C(t, D, Q, W) = ((0, D - e^Q - W)^+)^2$$
$$= (\max(0, D - Y - W))^2 \tag{6.12}$$

which satisfies law (5.9) on r_W.

The optimal discharge $W_t^*(Q_i, S_j)$ at time t, flow level Q_i and stocklevel $S_j (1 \le i, j \le 21)$ is obtained by (5.10), (5.11) and (5.12) on the basis of the calculation of $u_{S_j}(t + \Delta t, Q_i)$. Supply-side marginal utility $u_{S_j}(t, Q_i)$ $(i, j = 1, 2, \cdots, 21)$ at each time t is calculated backward on the basis of (5.16), (5.17) and an appropriately assumed

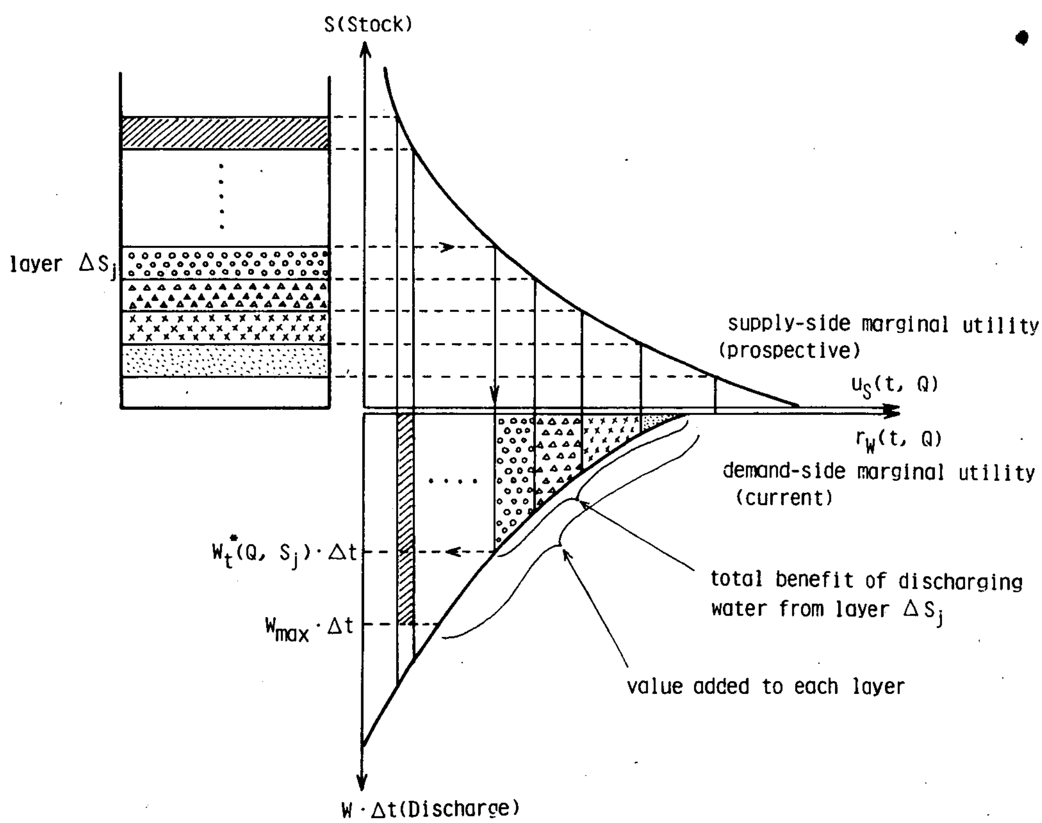

Fig. 6.3 Incremental value(value added) Δu_S of marginal utility u_S.

final marginal utility $u_{S_j}(T_1, Q_i)$ $(i,j = 1, 2, \cdots, 21)$ at time T_1. This final utility should be eveluated by solving the similar optimal (discharge) problem in the latter period i.e. from T_1 to T_2: the 31st of December. We will presume, however, several reasonable value of $u_{S_j}(T_1, Q_i)$ and analyze the effect of different setting of the value on the result , i.e. the optimal control W_t^* and marginal utility u_S.

It is interesting to note that incremental value (value added) Δu_S to the marginal utility given by (5.17) can be illustrated in Fig. 6.3. Namely, the total benefit of discharging (optimal amount of) water from layer $\Delta S_j \triangleq [S_{j-1}, S_j]$ in the reservoir is distributed back (returned) to each layer which is below or equal to ΔS_j according to its contribution to the benefit. The contribution of layer ΔS_k is

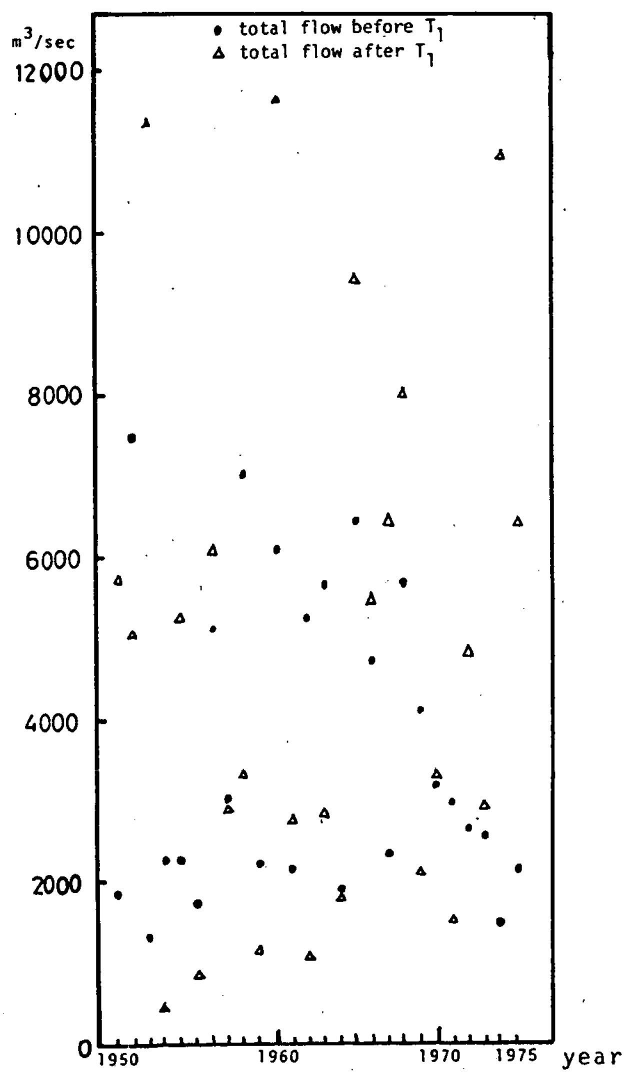

Fig. 6.4 Relation between total amount of flow before T_1 and that after T_1

evaluated as the difference in benefit between discharging water from ΔS_k and that from ΔS_{k-1} as shown in the figure. That is, the contribution of a layer is evaluated as the relative benefit of having water in the layer, and the layer increases its marginal utility with this amount of contribution as given by (5.16).

As shown earlier in Fig. 1.1 and also in Fig. 6.4, the behavior of the natural flow after T_1 (Sept. 30th) and that before T_1 have no significant interrelationships with each other. Hence, we can reasonably assume that the final marginal utility $u_S(T, Q)$ of stock S under flow livel Q at T_1 is independent of Q, i.e.,

$$u_{S_j}(T_1, Q_i) = \tilde{u}(S_j) \qquad \text{for} \quad i, j = 1, 2, \cdots, 21 \qquad (6.12)$$

Suppose that we can predict beforehand natural flow Y and demand D in the forthcoming control period $[T_1, T_2]$. Then as can be readily seen from (5.6) and (6.11), the marginal utility $\tilde{u}(S)$ can be approximated as a linear (monotonically decreasing) function of S with inclimation which is dependent on the total supply-demand gap in the forthcoming period. We presume eight different setting of $\tilde{u}(S_j)$ for stock level S_j $(j = 1, 2, \cdots, 21)$ as given in the sequel (cf. Fig. 6.5).

case 1 - 1 :	$\tilde{u}(S_j)$	$= 6600 - 600 \cdot j$	for	$1 \le j \le 10$
		$= 0$	for	$11 \le j \le 21$
case 1 - 2 :	"	$= 3300 - 300 \cdot j$		"
		$= 0$		"
case 1 - 3 :	"	$= 1650 - 150 \cdot j$		"
		$= 0$		"
case 1 - 4 :	"	$= 825 - 75 \cdot j$		"
		$= 0$		"
case 2 - 1 :	$\tilde{u}(S_j)$	$= 6400 - 400 \cdot j$	for	$1 \le j \le 15$
		$= 0$	for	$16 \le j \le 21$
case 2 - 2 :	"	$= 3200 - 200 \cdot j$		"
		$= 0$		"

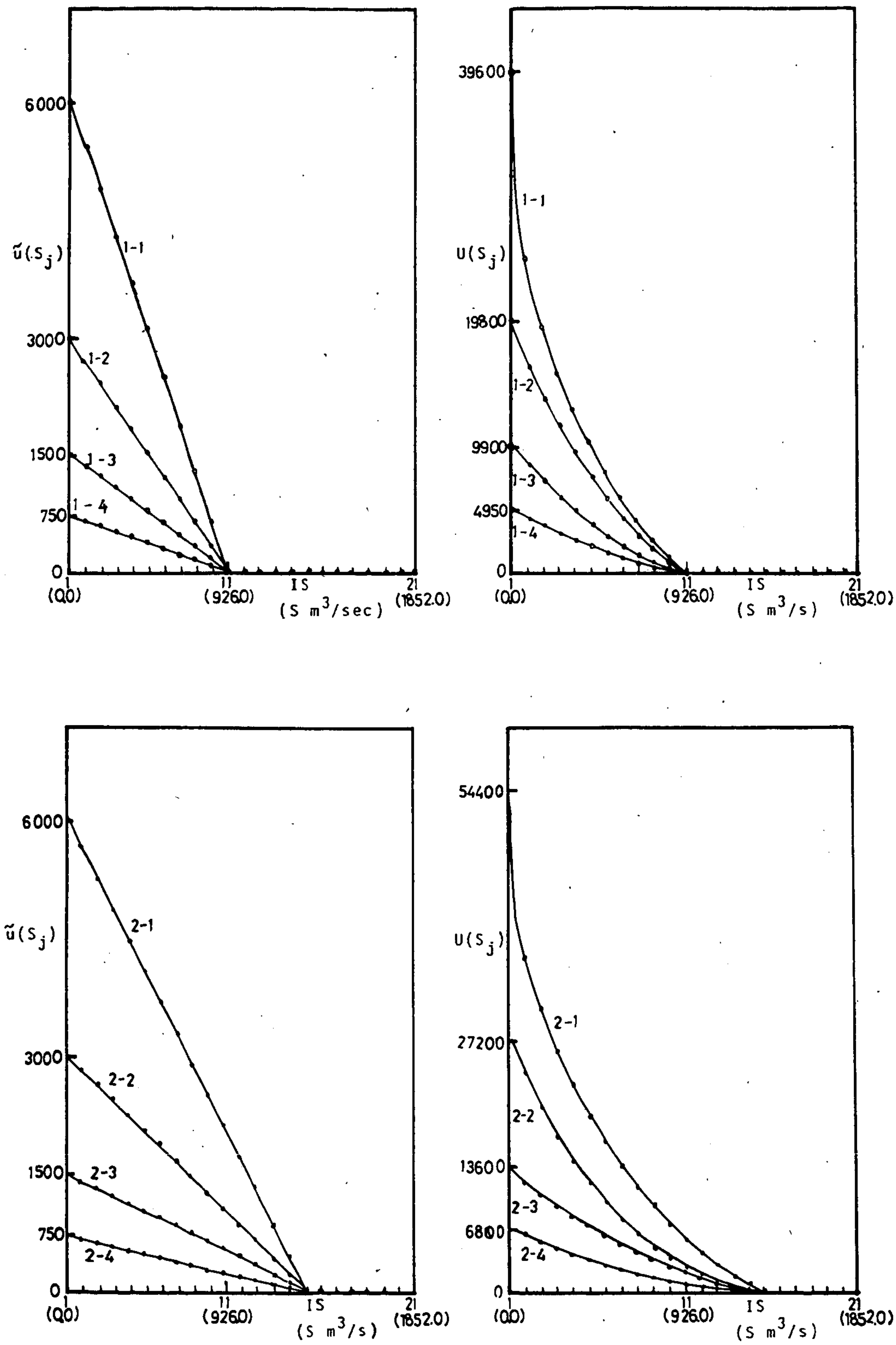

Fig. 6.5 Various setting of the final marginal utility u_S at time T_1.

case 2 - 3 :	"	$= 1600 - 100 \cdot j$	"	
		$= 0$	"	
case 2 - 4 :	"	$= 800 - 50 \cdot j$	"	
		$= 0$	"	
				(6.13)

In the upper four cases, it is assumed that half of maximum stock S_{max} is unnecessary for the discharge in the forthcoming period $[T_1, T_2]$, i.e., it is needless to store more than half of S_{max} at the final time T_1. In the lower four cases, one third of S_{max} is assumed to be unnecessary in the period. High slope in the figure corresponds to presuming large supply-demand gap and low slope corresponds to small gap.

By presuming a sample process $\{Y_t\}$ of natural flow shown in Fig. 6.6, we examine the dependence of the optimal discharge W_t^* on

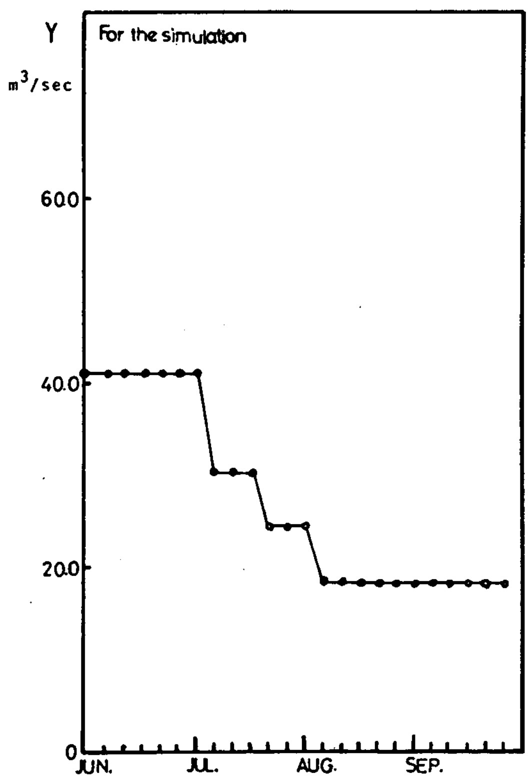

Fig. 6.6 A sample process of natural flow Y_t.

supply-demand gap $G_t \triangleq D_t - Y_t$. Fig. 6.7 and 6.8 show the result; W_t^* behaves quite coherently with G_t, i.e., the optimal discharge

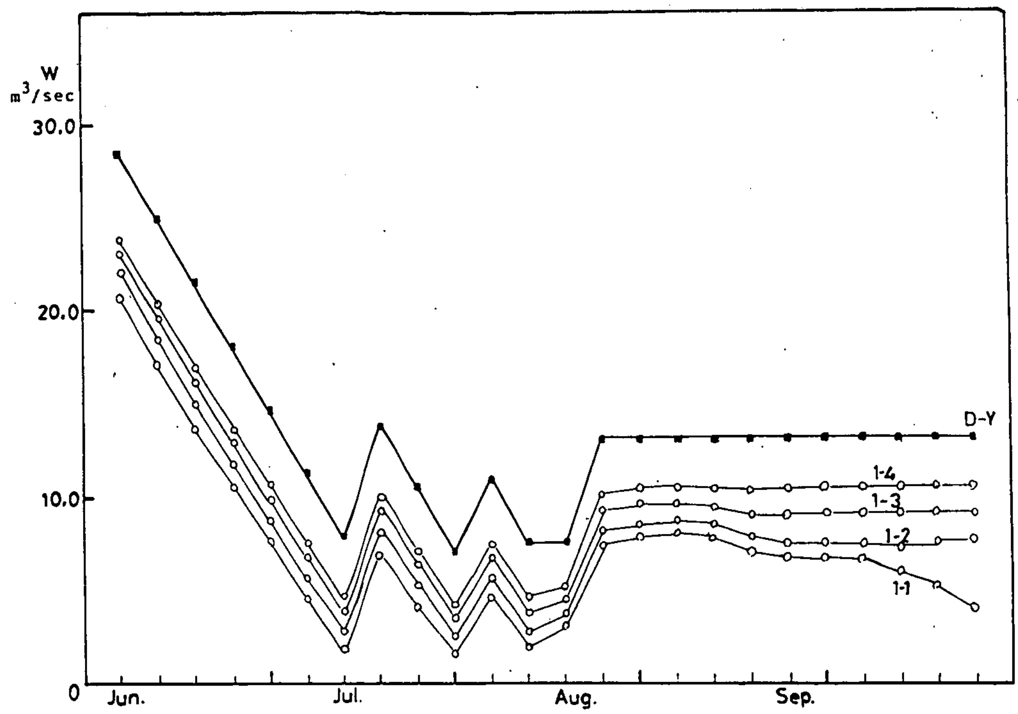

Fig. 6.7 Dependence of the optimal discharge W^* upon supply-demand gap D - Y for cases 1-1 ~ 1-4.

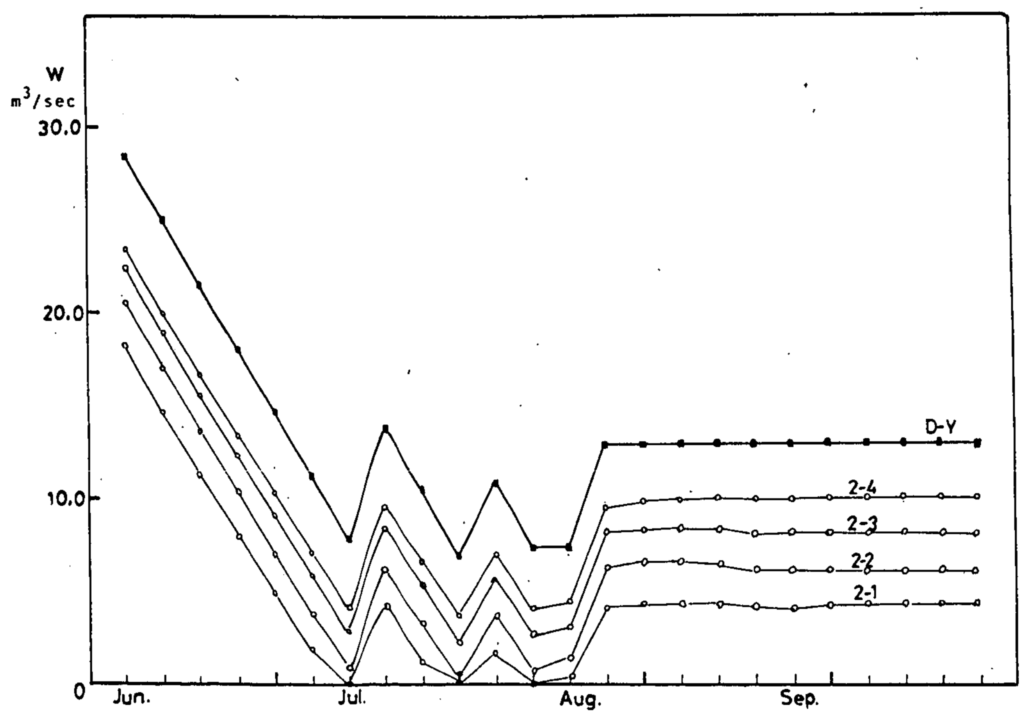

Fig. 6.8 Dependence of the optimal discharge W^* upon supply-demand gap D - Y for cases 2-1 ~ 2-4.

policy is to maintain the resultant gap $G_t - W_t$ after discharge as contant in order that integrated cost function $V = \int((G_t - W_t)^+)^2 dt$ be minimum. The resultant gap depends on the selection of the cases in (6.13), i.e., on the estimation of supply-demand gap (before discharge) in the forthcoming period. The final stock in cases 1-1 ~ 2-4 are 821.8, 678.2, 539.5, 421.9, 1099.6, 880.1, 643.5 and 470.0($m^3/s \times$ day).

Fig 6.9 - 6.12 show the evolution of marginal utility $u_S(t, Q)$ for fixed Q(Y = 18.4[m^3/sec]) according to the calculation backward on the basis of the initial setting $\tilde{u}(S)$ of u_S given by (6.12) and renewal formula of u_S, (5.16) and (5.17). The slope of the curves is very steep for low stock level S and early time t, i.e., for small s and small t. That is, if we have to discharge for a long period of time and have only small amount of stock, then the optimal discharge become extremely small.. Also, it should be noted that the difference in the setting of $u_S(T, Q)$ as given by (6.12) gradually looses its effect on $u_S(t, Q)$ as backward calculation going on; $u_S(t, Q)$'s become approximately the same for small t for cases 1-1 ~ 2-4.

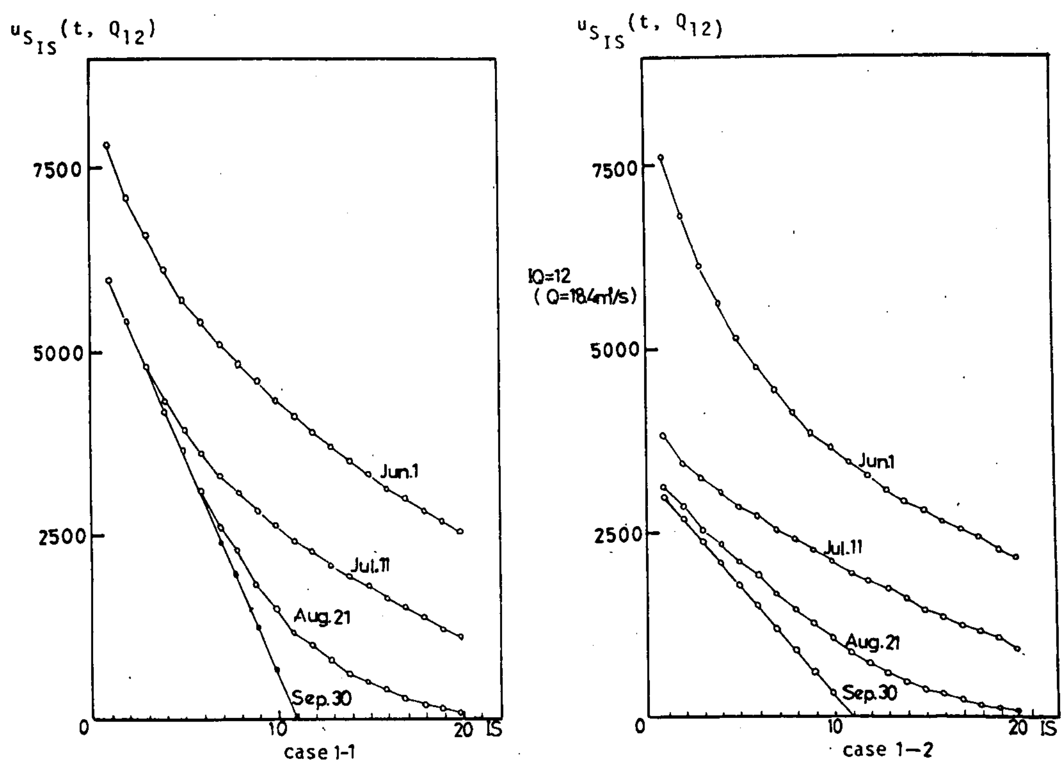

Fig. 6.9. Evolution of marginal utility u_S under cases 1-1 and 1-2.

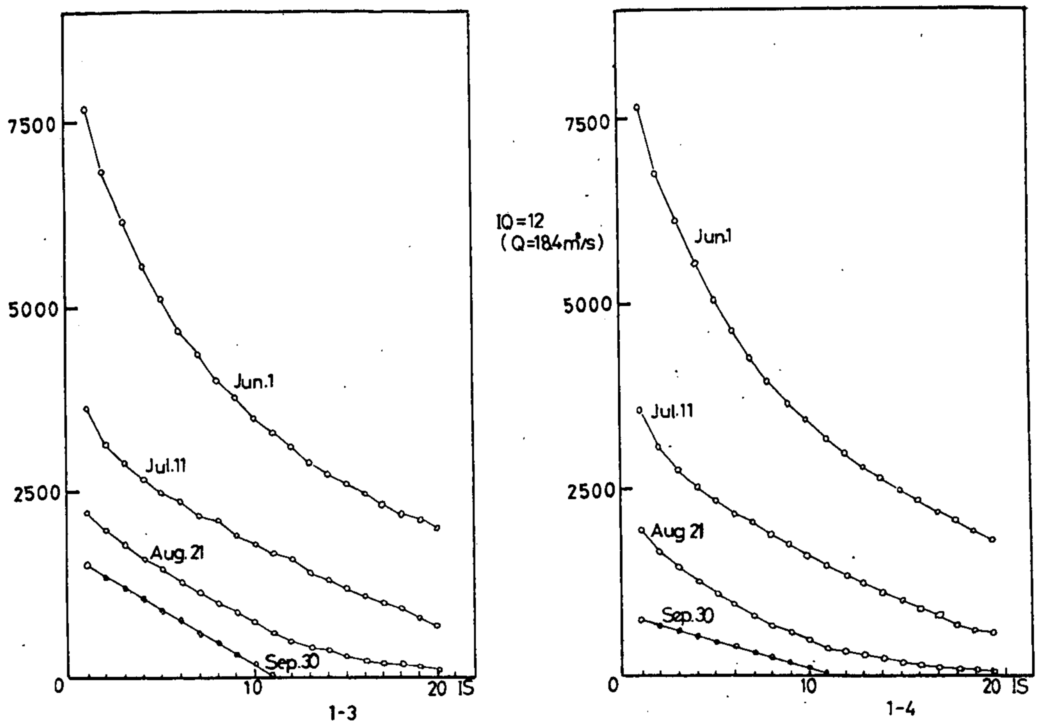

Fig. 6.10 Evolution of marginal utility u_S under cases 1-3 and 1-4.

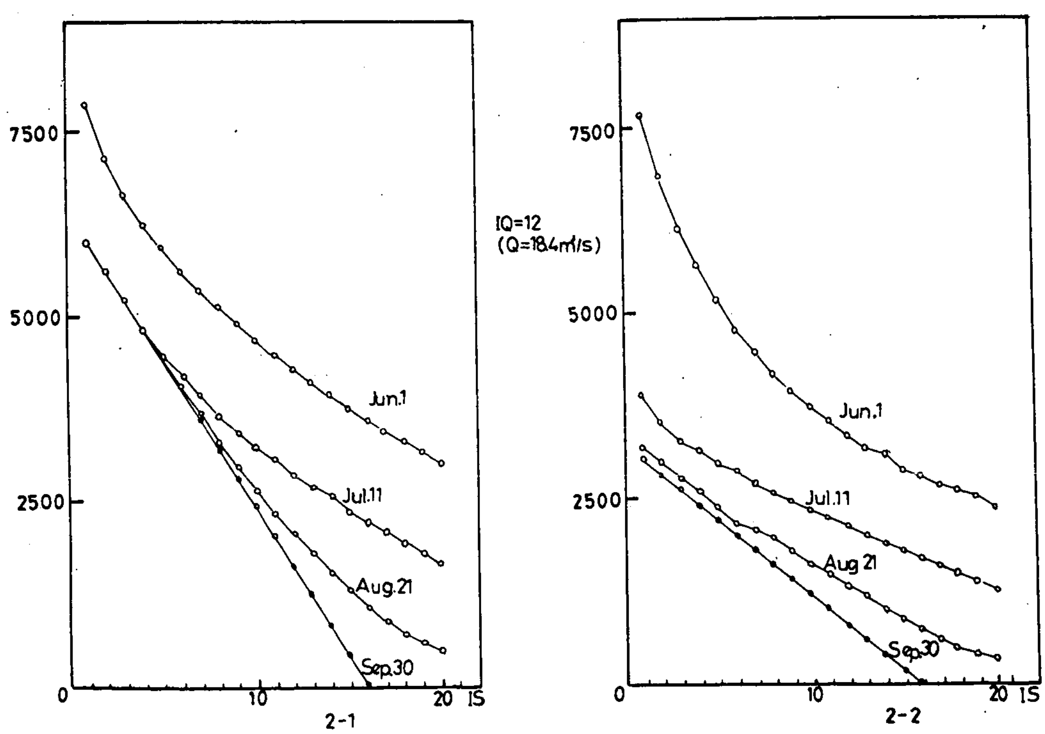

Fig. 6.11 Evolution of marginal utility u_S under cases 2-1 and 2-2.

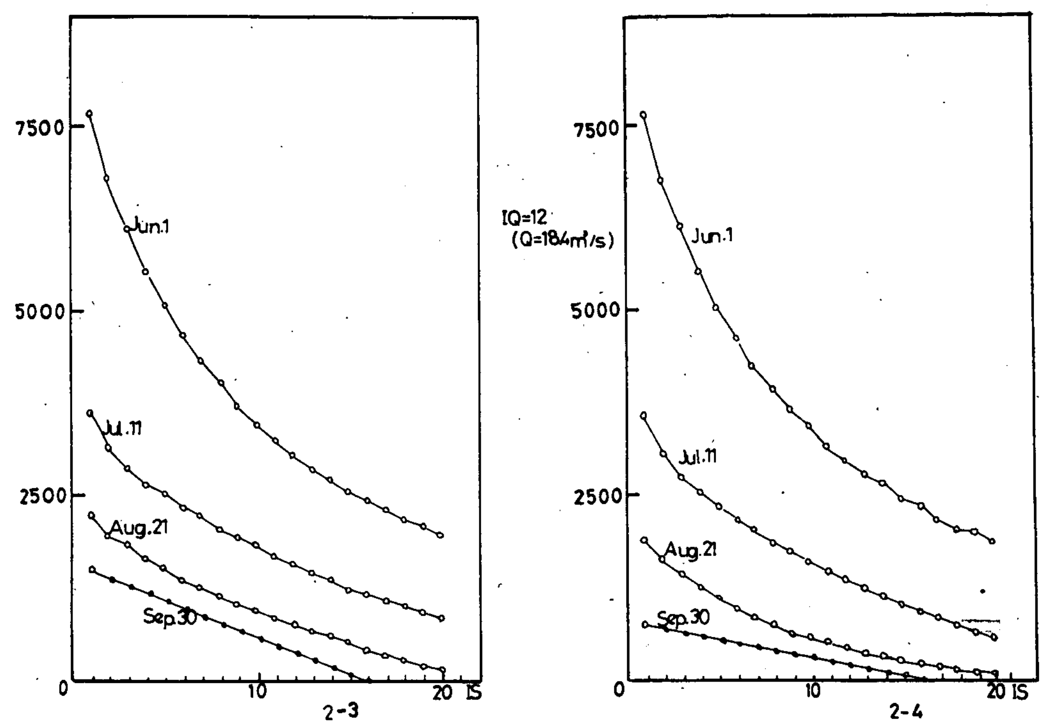

Fig. 6.12 Evolution of marginal utility u_S under cases 2-3 and 2-4.

Fig. 6.13 ~ 6.16 show the relation of marginal utility $u_S(t,Q)$ with natural flow Q for four different time instants (June 1st, July 11th, Aug. 21st, Sept. 30th) under fixed stock level S = 1296.4 (m^3/ sec × 1day). It is quite interesting to note that $u_S(t, Q)$ becomes more and more dependent on flow level Q along with its backward caluculation, while it is independent of Q at the initial setting, i.e. $u_S(T_1, Q)$ is independent of Q as given by (6.12). Also, it should be noted that the curve of $u_S(t, Q)$ versus Q is single-peaked. If we have enough natural flow (large Q), we need no discharge. Thus, it is obvious that u_S has small value under large Q. The small-valuedness of u_S at low flow level (at large supply-demand gap G_t) is interpreted as being due to the fact that the choice of discharge policy has no significant effect on the integrated cost $V = \sum(G_t - W_t)^2\Delta t$ because of stock S_{max} being far short of meeting the gap, i.e., like throwing water on thirsty soil.

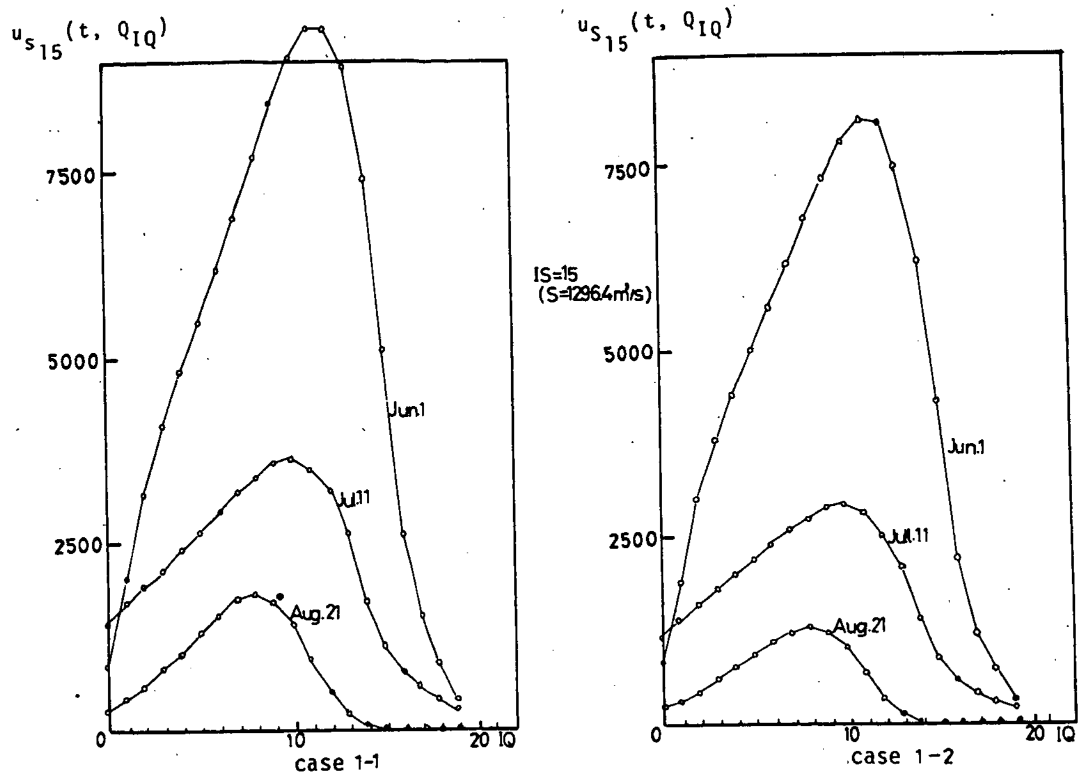

Fig. 6.13 Relation of marginal utility u_S with natural flow Q in cases 1-1 and 1-2.

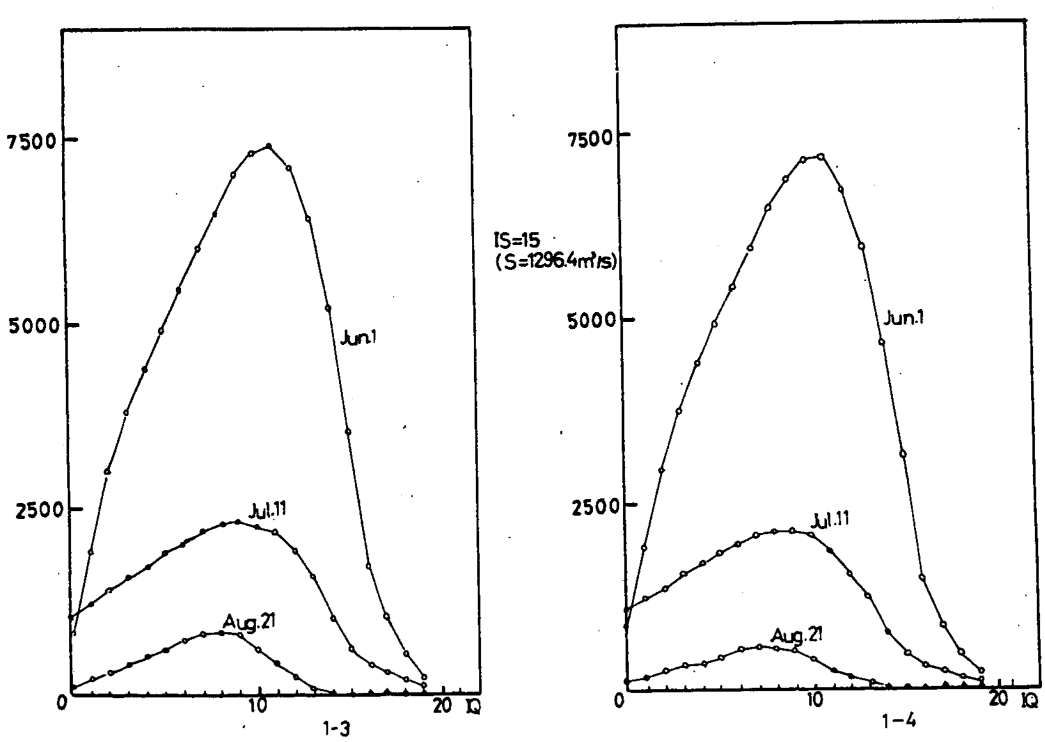

Fig. 6.14 Relation of marginal utility u_S with natural flow Q in cases 1-3 and 1-4.

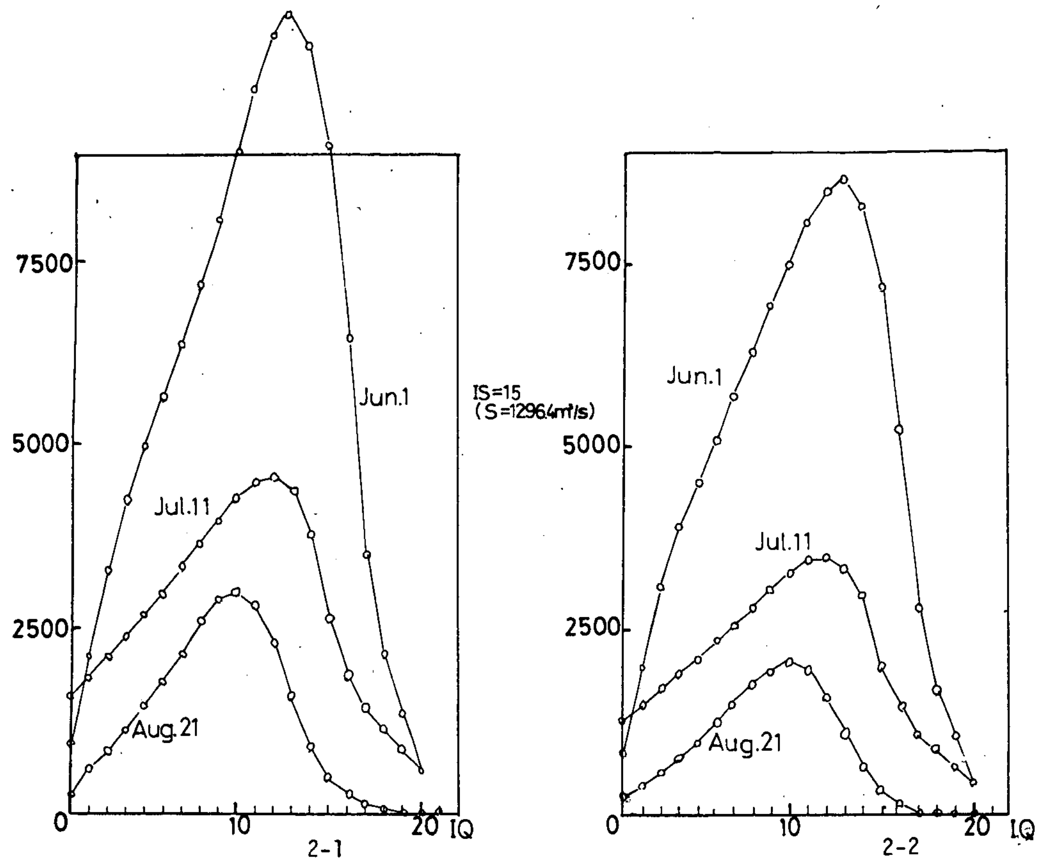

Fig. 6.15 Relation of marginal utility u_S with natural flow Q in cases 2-1 and 2-2.

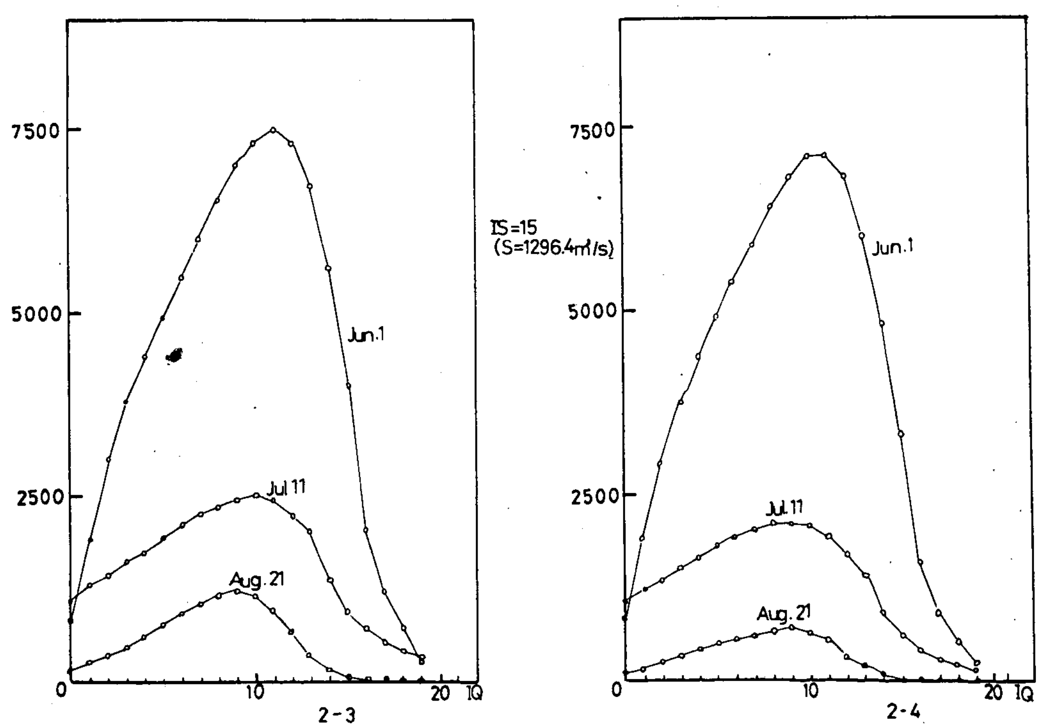

Fig. 6.16 Relation of marginal utility u_S with natural flow Q in cases 2-3 and 2-4.

Fig. 6.17 ~ 6.20 represent the equi-discharge curve (curve of stock level yielding constant amount of discharge versus time t) for constant demand D and constant natural flow Q, i.e. for constant supply-demand gap G. As mentioned in section 5, an equi-discharge curve coincides with an equi-marginal utility curve. The irregularity in the curves is due to the statistical nature of natural flow which is reflected in the regression coefficient α and β given in setion 3. The convexity of the curves in August reflects the genaral tendency of natural flow that it decreases in that period requesting more discharge than other period.

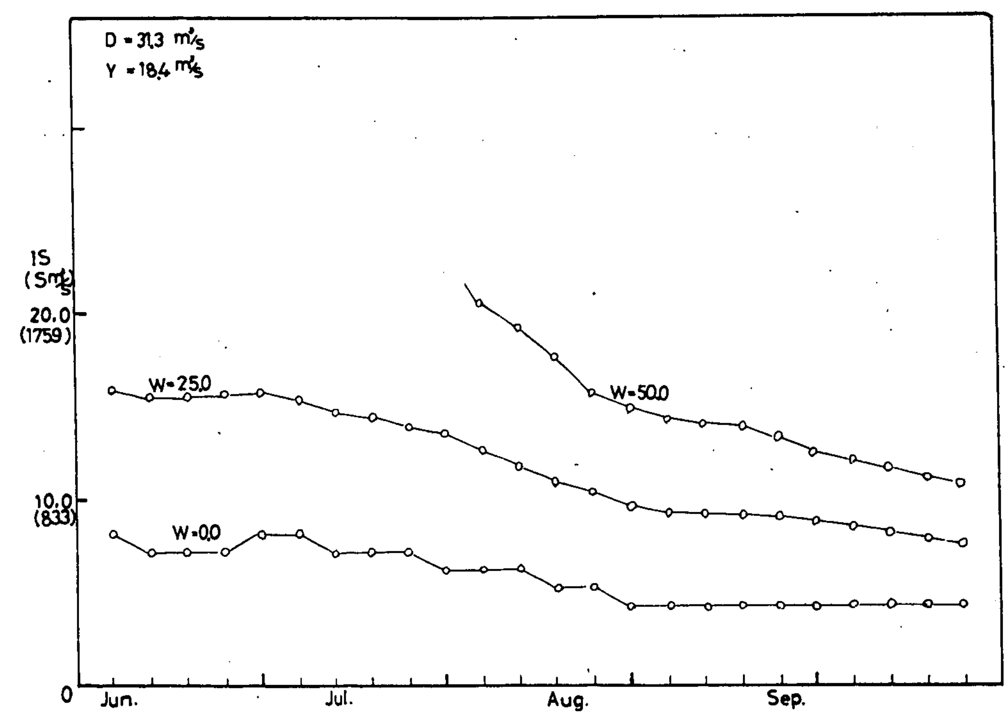

Fig. 6.17 Equi-discharge curve under constant demand D = 31.3(m^3/sec) and constant natural flow Y = 18.4(m^3/sec).

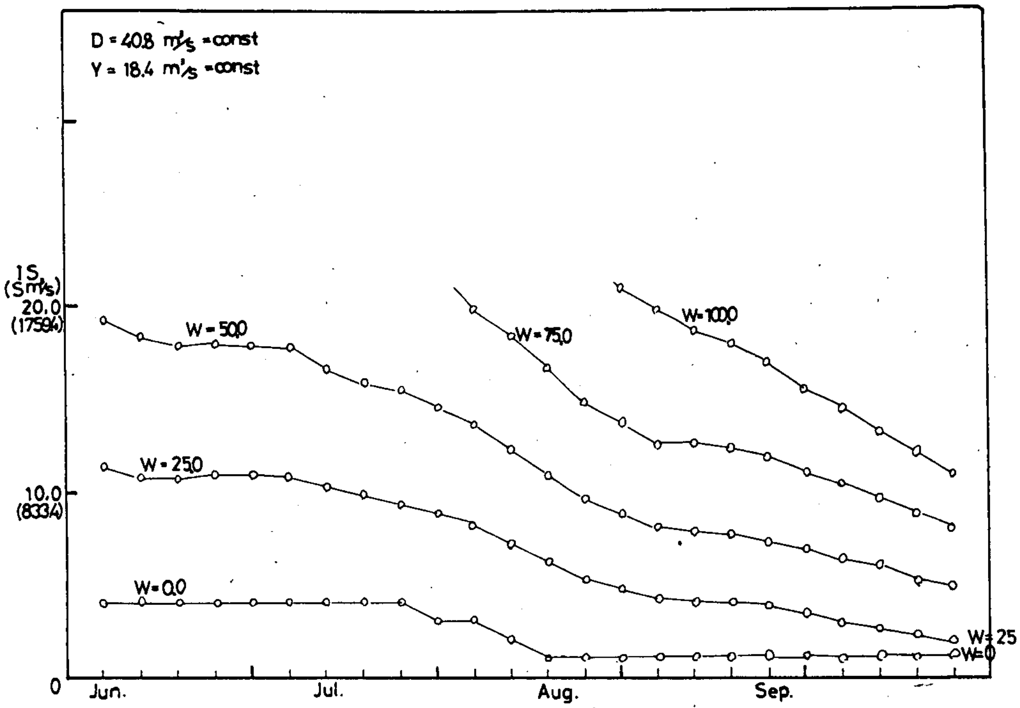

Fig. 6.18 Equi-discharge curve under constant demand D = 40.8 and constant natural flow Y = 18.4.

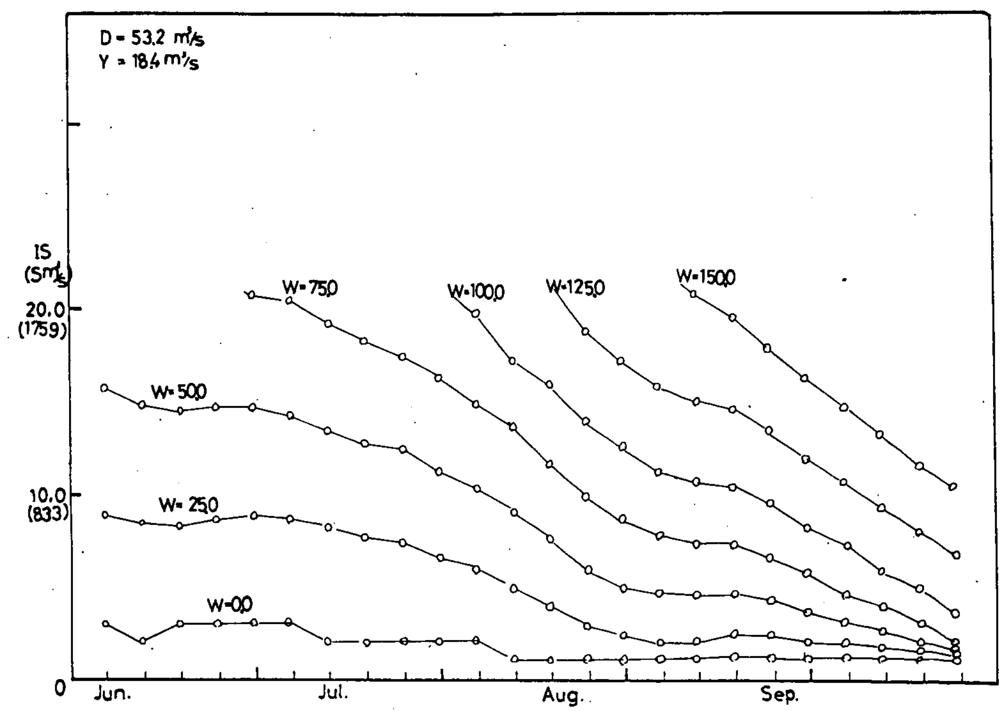

Fig. 6.19 Equi-discharge curve under constant demand D = 53.2 and constant natural flow Y = 18.4.

7. CONCLUSION

According to the accumulation of population and industrial activities, the control problem of water resources is one of the urgent urban problems today. This problem is, of course, concerned with a lot of factors in a very complicated manner. Through various assumptions, we have simplified the control problem of the dam, le barrage réservoir Seine, and arrived at the optimal discharge problem discussed above. Hence, the remainning part of the study is to examine the validity of theese assumptions.

Among them, it is quite necessary to examine the behavior of natural flow and that of demand in the latter subperiod $[T_1, T_2]$, i.e., from Oct. 1st to Dec. 31st, and then to clarify which of settings 1-1 ~ 2-4 is the most appropriate. Another important study is to examine the validity of logarithmic diffusion process model of natural flow $\{Y_t\}$; it should be noted, however, that if $\{Y_t\}$ is modelled in a more complicated way, the computation of optimal discharge needs much elaboration and necessitates a large amount of data in order to obtain accurate model and solution.

Being the identification of natural flow $\{Y_t\}$ based on its logarithmic value, $\ln Y_t$, the accuracy of the solution, the optimal discharge, is low under large Y_t. The accuracy will be recovered by introducing non-uniform discretization of Y-scale. Also, it should be noted that, while in this research we have assumed deterministic demand process, the assumption is unnecessary; the factor which determines the solution is not demand process $\{D_t\}$ but the process of supply-demand gap $G_t = D_t - Y_t$. Namely, we only have to identify stochastic process $\{G_t\}$ (neither $\{D_t\}$ nor $\{Y_t\}$).

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